

Dynamical laws of superenergy in General Relativity

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Abstract. The Bel and Bel-Robinson tensors were introduced nearly fifty years ago in an attempt to generalize to gravitation the energy-momentum tensor of electromagnetism. This generalization was successful from the mathematical point of view because these tensors share mathematical properties which are remarkably similar to those of the energy-momentum tensor of electromagnetism. However, the physical role of these tensors in General Relativity has remained obscure and no interpretation has achieved wide acceptance. In principle, they cannot represent *energy* and the term *superenergy* has been coined for the hypothetical physical magnitude lying behind them. In this work we try to shed light on the true physical meaning of *superenergy* by following the same procedure which enables us to give an interpretation of the electromagnetic energy. This procedure consists in performing an orthogonal splitting of the Bel and Bel-Robinson tensors and analysing the different parts resulting from the splitting. In the electromagnetic case such splitting gives rise to the electromagnetic *energy density*, the Poynting vector and the electromagnetic stress tensor, each of them having a precise physical interpretation which is deduced from the *dynamical laws* of electromagnetism (Poynting theorem). The full orthogonal splitting of the Bel and Bel-Robinson tensors is more complex but, as expected, similarities with electromagnetism are present. Also the covariant divergence of the Bel tensor is analogous to the covariant divergence of the electromagnetic energy-momentum tensor and the orthogonal splitting of the former is found. The ensuing *equations* are to the superenergy what the Poynting theorem is to electromagnetism. Some consequences of these *dynamical laws of superenergy* are explored, among them the possibility of defining *superenergy radiative states* for the gravitational field.

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1. Introduction

General Relativity is, in some aspects, a peculiar theory. In it the spacetime itself is part of the degrees of freedom and this fact brings to General Relativity some complications not present in other theories where the fields are set in a fixed spacetime background. One of these complications is the impossibility of defining a local invariant concept of *gravitational energy density*. The accepted argument to sustain this assertion relies on the equivalence principle. The consequence of this is that any geometric object representing gravitational “energy-momentum” can always be set to zero in a suitable coordinate system or frame and this property cannot be fulfilled by a tensor. Only a *pseudo-tensor* can accomplish this task but gravitational energy-momentum pseudo-tensors are not unequivocally defined because, by the very nature of a pseudo-tensor, they are always tied to a given frame or coordinate system. The use

of a pseudo-tensor makes it very difficult to address problems such as the calculation of the gravitational energy radiated by a source.

Different approaches to the “gravitational energy problem” in General Relativity have been provided along the years and no general formalism has emerged (although formalisms tailored for particular important cases do exist). One of these approaches seeks to enhance the formal similarities between electromagnetism and gravitation in order to find a replacement for the missing “gravitational energy-momentum tensor”. The idea is to take the electromagnetic energy-momentum tensor and translate it into a gravitational counterpart by somehow replacing the Faraday tensor with the Riemann tensor in the expression giving the energy-momentum tensor for electromagnetism. This translation is by no means straightforward due to the different nature of the Riemann and Faraday tensors but it can certainly be accomplished. The result of this translation is a four index tensor quadratic in the Riemann tensor which was first found by Bel [3]. The Bel tensor has mathematical properties which are remarkably similar to the electromagnetic energy-momentum tensor (see theorem 4.1 for a summary). An important particular case arises if we replace in the definition of the Bel tensor the Riemann with the Weyl tensor to give the Bel-Robinson tensor [2].

From the above considerations it is clear that the Bel tensor will represent a quantity which is different from energy. This new quantity was called “superenergy” by Bel and its status in General Relativity has been subject to much debate and no widely accepted conclusions have been reached. A simple dimensional analysis shows that in geometrized units the physical dimension of superenergy is L^{-4} where L represents length. Another important property is the tensorial character of superenergy. This means that if we work with *gravitational superenergy* instead of *gravitational energy* we can avoid all the technical complications arising when one works with pseudotensors. One of the main goals of this paper is to show what the consequences are of considering superenergy as a measurable physical quantity on its own. This means that we are not concerned in this work with the possible relationship between superenergy and other quantities with dimensions of energy.

In order to carry out our program we need to find the orthogonal splitting with respect to an observer of the Bel tensor (so we will be able to explain what the observer obtains when measuring superenergy) and we need to find the variation of the different parts of the orthogonal splitting along the observer’s path. The outcome of this last part is a set of equations which we call *the dynamical laws of superenergy* and they are the most important result of this paper.

We may examine at this point what the above procedure yields in the case of electromagnetism. In this case we are working with a quantity with dimensions of energy instead of dimensions of superenergy but this is now of no relevance. The different parts resulting from the orthogonal splitting of the electromagnetic energy-momentum tensor are the electromagnetic energy density, the Poynting vector and the electromagnetic stress-tensor. The utility of each of these parts is explained in basic electrodynamics textbooks. The *dynamical laws of electromagnetic energy* are contained in the Poynting theorem and it is through this theorem that the electromagnetic energy density and the Poynting vector gain their full physical meaning as measurable quantities. The Poynting theorem is nothing less than the orthogonal splitting of the covariant divergence of the electromagnetic energy-momentum tensor. The parts of this splitting are the variation of the electromagnetic energy density and the Poynting vector along the observer’s path. The Poynting theorem enables us to draw conclusions as important as the characterization of

radiative electromagnetic fields or the expression for the total force acting on an electromagnetic system.

In General Relativity we may consider the expression for the covariant divergence of the Bel tensor as the gravitational counterpart of the covariant divergence of the energy momentum tensor of electromagnetism. Therefore if we perform the orthogonal splitting of the former we will obtain a set of equations which can be regarded as the counterpart of the Poynting theorem. As mentioned before these equations are the dynamical laws of superenergy and they are far more complex than electromagnetism's Poynting theorem. However, we can still follow the same procedure as in electromagnetism to draw some conclusions and, for example, we can decide in a covariant way when a gravitational system is radiating superenergy (intrinsic superenergy radiative state). This was already attempted by Bel in the late fifties but since the full set of dynamical laws of superenergy was not available, Bel's result does not apply to cases that are sufficiently general.

The paper is organized as follows: in section 2 we review the notation and the essential concepts of orthogonal splittings. In section 3 we find the orthogonal splitting of the covariant divergence of the electromagnetic energy-momentum tensor in a general spacetime (theorem 3.2). This is the complete version of the classical Poynting theorem and some of its consequences are discussed. In section 4 we present the Bel and Bel-Robinson tensors and their essential mathematical properties are summarized in theorem 4.1. Section 5 contains the orthogonal splitting of the Bel-Robinson tensor and we study the basic mathematical properties of the different parts of the orthogonal splitting. Since these parts are expressed in terms of the electric and magnetic parts of the Weyl tensor, we can obtain particular *canonical forms* valid for some Petrov types (subsection 5.1). Section 6 is devoted to the orthogonal splitting of the Bel tensor and section 7 contains the main result of this paper which is theorem 7.1. This theorem spells out the different parts of the orthogonal decomposition of the covariant divergence of the Bel tensor (see equation (4.17)) which as explained above are the dynamical laws of superenergy. In section 8 we study the radiation of superenergy from a general point of view. To that end the definition of an *intrinsic superenergy radiative state* is put forward (definition 8.3).

The main results of this paper rely on heavy tensor calculations which can only be carried out with the aid of a computer algebra system. All the calculations of this paper have been undertaken with the computer program *xAct* [33]. *xAct* is a suite of MATHEMATICA packages devised to perform calculations in General Relativity and Differential Geometry. Among the many features of the *xAct* system we stress its ability to canonicalize tensor expressions by means of powerful algorithms based on permutation group theory (package *xPerm*), the excellent implementation of tensor calculus (package *xTensor*) and the possibility of working with frames and tensor components (package *xCoba*). In appendix A we provide further details about how *xAct* has been used in this paper. Currently, no other computer algebra system, either free or commercial, has the capabilities to perform the calculations needed in this paper.

2. The orthogonal splitting

We start by introducing the basic notation and conventions which will be adopted in this paper. We shall work in a four dimensional smooth Lorentzian manifold V which we will call *spacetime*. The abstract index notation is followed throughout to denote

tensors on V with Latin lowercase letters reserved for the abstract indices. We use bold typeface for component indices. Round (square) brackets enclosing indices denote index symmetrization (antisymmetrization). Unless otherwise stated all tensors are assumed smooth and defined globally on V . The metric tensor is g_{ab} and our signature convention is $(-, +, +, +)$. This metric is used to raise and lower indices in the usual way. Associated with the metric is the *volume element* which we denote by η_{abcd} . The Levi-Civita connection compatible with g_{ab} is the only affine connection ∇_a satisfying $\nabla_a g_{bc} = 0$ and our convention for the curvature tensor of this connection is fixed by the Ricci identity

$$\nabla_a \nabla_b X^c - \nabla_b \nabla_a X^c = X^d R_{bad}{}^c.$$

The Ricci tensor and the scalar curvature are $R_{bd} \equiv R_{bad}{}^a$ and $R \equiv R_a{}^a$ respectively. From these, the Einstein tensor is defined by the familiar formula $G_{ab} \equiv R_{ab} - Rg_{ab}/2$. The Lie derivative with respect to any vector field X^a is the differential operator \mathcal{L}_X . Geometrized units with $8\pi G = c = 1$ are used unless otherwise stated. The end of a proof is marked with \square .

Specially important for us are unit timelike vector fields. For any such vector field, the family of its integral curves defines a *timelike congruence* or *observer set*. This unit timelike vector field enables us to perform an *orthogonal splitting* (also called *3+1 decomposition*) of any tensor on V . The orthogonal splitting lies at the basis of many studies and formalisms in General Relativity and has been extensively studied in the literature but since it will be used in this work many times we now review its essentials (good accounts can be found in [17, 32]). Let n^a be any vector field with $n_a n^a = -1$ and define the *spatial metric* h_{ab} by

$$h_{ab} \equiv g_{ab} + n_a n_b, \quad h_{ab} h^b{}_c = h_{ac}, \quad h^a{}_a = 3. \quad (2.1)$$

The tensor h_{ab} has the properties of an orthogonal projector. We shall call a covariant tensor $T_{a_1 \dots a_m}$ *spatial* with respect to h_{ab} if it is invariant under $h^a{}_b$ i.e. if

$$h^{a_1}{}_{b_1} \dots h^{a_m}{}_{b_m} T_{a_1 \dots a_m} = T_{b_1 \dots b_m},$$

with the obvious generalization for any mixed tensor. This property implies that the inner contraction of n^a with $T_{a_1 \dots a_m}$ (taken on any index) vanishes. We introduce next the orthogonal projection operator defined by

$$P_h(L_{a_1 \dots a_m}) \equiv h^{s_1}{}_{a_1} \dots h^{s_m}{}_{a_m} L_{s_1 \dots s_m}, \quad (2.2)$$

where $L_{a_1 \dots a_m}$ is an arbitrary tensor. Clearly $P_h(L_{a_1 \dots a_m})$ is a spatial tensor. Another definition which we need is the *generalized inner contraction* of the tensor $L_{a_1 \dots a_m}$ with the unit normal which is given by

$$n^J(L_{a_1 \dots a_m}) \equiv n^{s_1} \dots n^{s_{\#J}} L_{\dots s_1 \dots s_2 \dots s_{\#J} \dots}.$$

Here J is an ordered subset of the set of abstract indices $\{a_1 \dots a_m\}$ and the dummies $\{s_1 \dots s_{\#J}\}$ are placed in those slots of L indicated by J . Therefore $n^J(L_{a_1 \dots a_m})$ has $m - \#J$ free indices given by the complement of J with respect to $\{a_1 \dots a_m\}$. Using the orthogonal projection operator and the generalized inner contraction we find that any tensor $L_{a_1 \dots a_m}$ can be written in the following way

$$L_{a_1 \dots a_m} = \sum_{J \in \mathcal{P}(\{a_1 \dots a_m\})} (-1)^{\#J} n_J P_h(n^J(L_{a_1 \dots a_m})), \quad (2.3)$$

where $\mathcal{P}(\{a_1 \dots a_m\})$ is the power set of $\{a_1 \dots a_m\}$ and

$$n_J \equiv n_{a_q} \dots n_{a_p}, \quad J = \{a_q, \dots, a_p\} \in \mathcal{P}(\{a_1 \dots a_m\}).$$

The right hand side of (2.3) is called the *orthogonal splitting* of $L_{a_1 \dots a_m}$ with respect to the unit normal n^a (we will just speak of orthogonal splitting of a tensor if the unit normal is understood). The orthogonal splitting given by (2.3) is unique and the set of spatial tensors $\{P_h(n^J(L_{a_1 \dots a_m}))\}$ contains all the information about $L_{a_1 \dots a_m}$. Equation (2.3) is just the traditional calculation of the orthogonal splitting of a tensor written in a short form. It is possible to study the orthogonal splitting of a general tensor in an alternative way if we regard it as a *r-fold form* (see [38, 39] for a precise explanation of this).

A trivial example of orthogonal splitting is that of the metric tensor itself which is obtained from the first expression in (2.1). Another important example of orthogonal splitting which is easily deduced from (2.3) is

$$\eta_{abcd} = -n_a \varepsilon_{bcd} + n_b \varepsilon_{acd} - n_c \varepsilon_{abd} + n_d \varepsilon_{abc},$$

where ε_{abc} is the *spatial volume element* and is defined by

$$\varepsilon_{abc} \equiv n^d \eta_{dabc}.$$

2.1. Kinematical quantities

As we explained above, the set of integral curves of n^a represents a family of observers. In physical applications it is important to introduce quantities describing the *relative motion* of each curve of the family and this is the role of the kinematical quantities. To define them we write down the orthogonal splitting of $\nabla_a n_b$ which is

$$\nabla_a n_b = -A_b n_a + \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \omega_{ab}. \quad (2.4)$$

The tensor A_b is the acceleration, the scalar θ is the expansion and σ_{ab} , ω_{ab} are the shear and the rotation respectively. From the previous equation it is easy to obtain expressions for the kinematical quantities in terms of n^a

$$A^b = n^a \nabla_a n^b, \quad \theta = \nabla_a n^a, \quad \omega_{ab} = h_{[a}^d h_{b]}^c \nabla_d n_c, \quad \sigma_{ab} = h_{(a}^d h_{b)}^c \nabla_d n_c - \frac{\theta}{3} h_{ab}. \quad (2.5)$$

Straightforward properties of the kinematical quantities are

$$\sigma_{(ab)} = \sigma_{ab}, \quad \omega_{[ab]} = \omega_{ab}, \quad \sigma^a_a = 0, \quad (2.6)$$

Sometimes the rotation is replaced by the *vorticity* which is defined as follows

$$\omega_a \equiv \frac{1}{2} \varepsilon_{abc} \omega^{bc} \Rightarrow \omega_{ab} = \varepsilon_{abc} \omega^c. \quad (2.7)$$

Each of the kinematical quantities has a precise interpretation which deals with the *relative motion* of the observers of the congruence (see e.g. [17, 21] for a more detailed description of these concepts).

2.2. Cattaneo operator

Another very important object, which is needed when working with orthogonal splittings is the *Cattaneo operator* also known as *spatial connection* [14]. If $L_{a_1 \dots a_m}$ is a covariant tensor then we define the linear operator

$$D_a L_{a_1 \dots a_m} \equiv P_h(\nabla_a L_{a_1 \dots a_m}), \quad (2.8)$$

with obvious definitions for contravariant and mixed tensors. The Cattaneo operator is not a linear connection on the spacetime manifold V because it does not satisfy the Leibnitz rule unless both factors of the product upon which D_a acts are spatial. From its definition, it is clear that $D_a T_{b_1 \dots b_m}$ is a spatial tensor. Important additional properties of the Cattaneo operator are

$$D_a h_{bc} = 0, \quad D_a D_b \varphi - D_b D_a \varphi = 2n^c \omega_{ab} \nabla_c \varphi = 2\omega_{ab} \mathcal{L}_n \varphi, \quad \varphi \in C^1(V). \quad (2.9)$$

The Cattaneo operator enables us to write in a compact form the orthogonal splitting of any expression involving derivatives. It is specially important in this work to find the orthogonal splitting of the covariant derivative of a spatial tensor. To illustrate how this works, let us study a particular simple example. Consider $\nabla_a L_b$, where L_b is an arbitrary spatial covector ($n^a L_a = 0$). In this case formula (2.3) yields

$$\nabla_a L_b = D_a L_b - n_a P_h(n^p \nabla_p L_b) - n_b P_h(n^p \nabla_p L_a) + n_a n_b n^p n^q \nabla_p L_q. \quad (2.10)$$

Next we use in this equation the relations

$$n^c \nabla_c L_a = \mathcal{L}_n L_a - L_c \nabla_a n^c, \quad n^p \nabla_a L_p = -L_p \nabla_a n^p, \quad (2.11)$$

and replace in (2.10) the covariant derivatives of the unit normal by the expression given in (2.4). After some manipulations equation (2.10) becomes

$$\begin{aligned} \nabla_a L_b = & -A^c L_c n_b n_a + \left(\frac{1}{3} L_b \theta - \mathcal{L}_n L_b + L^c (\sigma_{bc} + \omega_{bc}) \right) n_a + D_a L_b + \\ & + n_b \left(\frac{1}{3} L_a \theta + L^c (\sigma_{ac} + \omega_{ac}) \right), \end{aligned} \quad (2.12)$$

which has the form of (2.3) and hence is the complete orthogonal splitting of $\nabla_a L_b$. Note that $\mathcal{L}_n L_a$ is a spatial covector if L_a is spatial, due to the property $\mathcal{L}_n n^a = 0$. The procedure followed to obtain (2.12) is easily generalized for the covariant derivative of any spatial tensor (see appendix A for more examples). This kind of calculation is extensively used in section 7.

3. Electromagnetism as a working example

As a preparation for the study which we are going to undertake of the gravitational field, we analyse first the case of electromagnetism. The electromagnetic field is described by an antisymmetric rank-2 tensor F_{ab} (the Faraday or electromagnetic tensor) which satisfies the *Maxwell equations*

$$\nabla_{[a} F_{bc]} = 0, \quad \nabla_a F^a_b = j_b, \quad (3.1)$$

where j^b is the *charge current* four vector (here we follow the Heaviside-Lorentz units system). A very important object in electromagnetic theory is the energy-momentum tensor of the electromagnetic field, given by

$$T_a^b = \frac{1}{2} (F_{ad} F^{bd} + F_{ad}^* F^{*bd}) = F_{ad} F^{bd} - \frac{1}{4} \delta^a_b F_{cd} F^{cd}. \quad (3.2)$$

In this formula F_{ab}^* is the Hodge dual of F_{ab} defined by

$$F_{ab}^* \equiv \frac{1}{2} \eta_{abcd} F^{cd}.$$

Theorem 3.1 *The tensor T_{ab} has the following properties*

- (i) $T_{(ab)} = T_{ab}$.
- (ii) T_{ab} always satisfies the dominant energy condition, namely, for any pair u^a, v^a of causal future-directed vector fields the inequality $T_{ab}u^av^b \geq 0$ holds.
- (iii) If the Maxwell equations hold then we have

$$\nabla_b T^b_a = F_a{}^b j_b, \quad \nabla_a j^a = 0. \quad (3.3)$$

Roughly speaking, the first equation of (3.3) tells us that the variation of the electromagnetic energy-momentum equals the work performed by the charge current and the second equation is the equation of charge conservation. An adequate understanding of these informal assertions can be achieved by finding the orthogonal splitting of (3.3). As an aside remark, we note that the first equation of (3.3) is not in general equivalent to Maxwell equations as is sometimes wrongly stated.

To find the orthogonal splitting of (3.3) we first need to find the orthogonal splitting of F_{ab} . Define the spatial tensors

$$E_a \equiv F_{ab}n^b, \quad B_a \equiv F_{ab}^*n^b \quad (3.4)$$

These are the electric and magnetic parts of the Faraday tensor and they characterize it completely. The orthogonal decomposition of the Faraday tensor in terms of E_a and B_a reads

$$F_{ab} = E_b n_a - E_a n_b - B_p \varepsilon_{ab}^p. \quad (3.5)$$

Using this expression, we can find the orthogonal splitting of the energy-momentum tensor T_{ab} which results in

$$\begin{aligned} T_{ab} &= U n_a n_b + 2P_{(a} n_{b)} + \mathcal{T}_{ab}, \\ U &\equiv \frac{1}{2}(E_a E^a + B_a B^a), \quad P_a \equiv \varepsilon_{abc} B^b E^c, \quad \mathcal{T}_{ab} \equiv U h_{ab} - E_a E_b - B_a B_b. \end{aligned} \quad (3.6)$$

Also, the orthogonal splitting of j^a is easily found yielding

$$j^a = \rho n^a - J^a, \quad \rho \equiv -j^a n_a, \quad J^a \equiv -h^{ac} j_c$$

where ρ is the charge density and J^a is the spatial charge current. Next we replace the decomposition of T_{ab} and j^a in (3.3) and calculate the orthogonal splitting of the resulting equations. To achieve this we need to find the orthogonal splitting of $\nabla_a U$, $\nabla_a P_b$ and $\nabla_a \mathcal{T}_{bc}$ which is done by using the appropriate generalizations of (2.12) (see the proof of theorem 7.1, theorem 7.2 in appendix A and especially equation (A.1)). The final result is presented next.

Theorem 3.2 *The following set of equations*

$$\nabla_b T^b_a = F_a{}^b j_b, \quad \nabla_a j^a = 0,$$

is equivalent to

$$\mathcal{L}_n U = -E^a J_a - 2A^a P_a - \frac{4}{3}U\theta - \mathcal{T}^{ab}\sigma_{ab} - D_a P^a, \quad (3.7)$$

$$\mathcal{L}_n P_a = -\varepsilon_{abc} B^b J^c + E_a \rho + 2\varepsilon_{abc} P^b \omega^c - P_a \theta - A^b (U h_{ab} + \mathcal{T}_{ab}) - D_b \mathcal{T}_a^b, \quad (3.8)$$

$$\mathcal{L}_n \rho = -A^a J_a + \theta \rho + D_a J^a. \quad (3.9)$$

Equations (3.7)-(3.8) are presented in basic electrodynamics books under the heading of the Poynting theorem and they reflect the transfer of energy-momentum in a system composed of charged particles and electromagnetic fields. Indeed, equations (3.7)-(3.8) provide the well-known physical interpretation of each of the quantities appearing in equation (3.6): U is the electromagnetic energy density, P^a is the Poynting vector and \mathcal{T}_{ab} is the stress tensor of the electromagnetic field (see e.g. [26] for detailed explanations about the role of each of these quantities).

We must note at this point that equations (3.7)-(3.8) are usually presented under the assumption that the spacetime is flat and n^a is chosen in such a way that all the kinematical quantities vanish. The resulting equations can be always obtained *locally* in a general spacetime if we recall that we can always construct a vector field n^a with the property that all its kinematical quantities vanish at a prescribed point (equivalence principle). Therefore, we deduce from these considerations that we can classify the terms which appear in (3.7)-(3.8) into two categories: those which contain kinematical quantities and those which do not. Terms which do not contain kinematical quantities can be regarded as representing *intrinsic* variations of energy or momentum (non-inertial terms) whereas terms affected by kinematical quantities can be thought of as depending on the observer n^a and we shall call them *inertial terms* in analogy to the inertial forces introduced in the study of accelerated systems in Newtonian physics. These considerations, although elementary, will play an important role in section 8.

3.1. Coupling of the vorticity and the Poynting vector

If the vector field n^a is hypersurface orthogonal then (3.7)-(3.8) assume simpler forms which can be found in different places in the literature [42]. The general form of (3.7) is written down in [32] but to the best of our knowledge equation (3.8) does not seem to be present in accessible references. Also some of the consequences of (3.8) do not appear to be widely known. To illustrate this fact, consider the inertial terms in (3.8). In ordinary units we find that the left hand side of (3.8) is the time variation of the momentum density and therefore the terms on the right hand side of (3.8) which are coupled to the kinematical quantities can be regarded as inertial forces. Indeed equation (3.8) can be interpreted as an equilibrium condition for an electromagnetic system which states that the sum of all (inertial and non inertial) forces acting on the system equals zero.

One of the inertial forces is given by $2\varepsilon_{abc} P^b \omega^c$ or in three-vector notation $2\vec{P} \times \vec{\omega}$ with “ \times ” representing the vector product. If we consider a gyroscope then we find that the vorticity ω^a is related to the angular velocity of the gyroscope. Therefore we deduce that an inertial force exists on an uncharged gyroscope when it is placed in a radiative electromagnetic field. In SI units this inertial force can be estimated by

$$\vec{F}_g \approx \frac{2V}{c^2} \vec{P} \times \vec{\omega}, \quad (3.10)$$

where V is the volume of the gyroscope. Thus we conclude that the flux of electromagnetic radiation produces an effect on a gyroscope. We must stress at this point that this is an observer dependent effect (as it should be because we are dealing with an inertial force) which manifests itself in the fact that the angular velocity of the gyroscope depends on the observer. An effect similar to this was pointed out in a particular case in [12] and this was latter confirmed in [22]. In the former reference it was shown that gyroscopes placed in the spacetime generated by a nonrotating charged magnetic dipole would precess. As an explanation of this result it was suggested that the Poynting vector could cause a measurable effect on a gyroscope's precession and it is conceivable that (3.10) is related to this effect in some way.

4. Gravitational equations and the Bel tensor

We start this section by reviewing the well-known formal analogy which exists between electromagnetism and gravitation. In this framework the Riemann tensor R_{abcd} is taken to be as the gravitational counterpart of the Faraday tensor F_{ab} and the role of the two Maxwell equations is played by the relations

$$\nabla_{[a} R_{bc]d} = 0, \quad \nabla_d R_{bpc}{}^d = \mathfrak{J}_{pbc}, \quad \mathfrak{J}_{efa} \equiv \nabla_e R_{af} - \nabla_f R_{ae} \quad (4.1)$$

The tensor \mathfrak{J}_{abc} is known as the *matter current* and can be regarded as the counterpart of the charge current four vector j^a . There is an important difference between electromagnetism and gravitation in that in the latter we have an extra set of conditions: the Einstein field equations

$$G_{ab} = \mathfrak{T}_{ab}. \quad (4.2)$$

Here the tensor \mathfrak{T}_{ab} is the energy-momentum tensor of the system and must be prescribed independently. Clearly any solution of the Einstein equations will be a solution of (4.1) but the converse need not be true. From (4.1) we derive the well-known relation (see e.g. [16])

$$\begin{aligned} \nabla_a \nabla^a R_{dcbp} = & \nabla_b \mathfrak{J}_{dcp} - \nabla_p \mathfrak{J}_{dcb} - 2R_{dc}{}^{ae} R_{bape} - R_b{}^a R_{dcpa} + R_{dcba} R_p{}^a - 2R_{cape} R_d{}^a{}^e{}_b + \\ & + 2R_{beca} R_d{}^a{}^e{}_p, \end{aligned} \quad (4.3)$$

which can be shown to be a hyperbolic equation for the Riemann tensor. A result due to Lichnerowicz [31] proves that if the Cauchy data of (4.3) satisfy (4.2) then so does the solution of the hyperbolic equation. Hence, with the provision imposed by the Lichnerowicz result, we can regard (4.1) and (4.2) as equivalent.

4.1. Orthogonal splitting of the Riemann tensor

The orthogonal splitting of the Riemann tensor was first studied in [4] and since then it has been used in many places. Define the left, right and double dual of Riemann tensor in the standard fashion

$${}^*R_{abcd} \equiv \frac{1}{2} \eta_{abpq} R^{pq}{}_{cd}, \quad R_{abcd}^* \equiv \frac{1}{2} \eta_{pqcd} R_{ab}{}^{pq}, \quad {}^*R_{abcd}^* = \frac{1}{2} \eta_{ab}{}^{pq} R_{pqcd}^*.$$

Next we introduce the following spatial tensors [4]

$$Y_{ac} \equiv R_{abcd} n^b n^d, \quad Z_{ac} \equiv {}^*R_{abcd} n^b n^d, \quad X_{ac} \equiv {}^*R_{abcd}^* n^b n^d \quad (4.4)$$

The symmetries of Riemann tensor entail the properties

$$X_{(ab)} = X_{ab}, \quad Y_{(ab)} = Y_{ab}, \quad Z_a{}^a = 0. \quad (4.5)$$

These tensors contain all the information in the Riemann tensor as is easily checked by a simple count of their total number of independent components. They also enable us to find the orthogonal splitting of the Riemann tensor which reads

$$R_{abcd} = 2n_c n_{[a} Y_{b]d} + 2h_{a[d} X_{c]b} + 2n_d n_{[b} Y_{a]c} + 2n_{[d} Z_{c]}^e \varepsilon_{abe} + 2n_{[b} Z_{a]}^e \varepsilon_{cde} + h_{bd} (h_{ac} X_e^e - X_{ac}) + h_{bc} (X_{ad} - h_{ad} X_e^e). \quad (4.6)$$

From this expression is easy to get the orthogonal splitting of the Ricci tensor which is

$$R_{ac} = Z^{db} \varepsilon_{cdb} n_a + n_c Y_d^d n_a - X_{ac} - Y_{ac} + n_c Z^{db} \varepsilon_{adb} + h_{ac} X_d^d. \quad (4.7)$$

The Weyl tensor C_{abcd} has the same algebraic properties as the Riemann tensor and in addition it is completely traceless. Therefore to find its orthogonal splitting we proceed along the same lines as with the Riemann tensor but using different names for the tensors introduced in (4.4). The precise correspondences are (in the next equation X_{ab} , Y_{ab} , Z_{ab} are defined as in (4.4) with the Riemann replaced by the Weyl tensor)

$$B_{ab} \equiv Z_{ab} = Z_{(ab)}, \quad E_{ab} \equiv Y_{ab} = -X_{ab}, \quad E_a^a = 0. \quad (4.8)$$

The tensors E_{ab} and B_{ab} are known as the electric and magnetic parts of the Weyl tensor and they completely characterize the former. Equation (4.6) becomes for the Weyl tensor

$$C_{abcd} = 2n_c n_{[a} E_{b]d} - 2h_{a[d} E_{c]b} + 2n_d n_{[b} E_{a]c} + 2n_{[d} B_{c]}^e \varepsilon_{abe} + 2n_{[b} B_{a]}^e \varepsilon_{cde} + 2h_{b[d} E_{c]a}. \quad (4.9)$$

4.2. Orthogonal splitting of the matter current

The orthogonal splitting of \mathfrak{J}_{abc} can be calculated if we insert in the last expression of (4.1) the orthogonal decomposition of the Ricci tensor (4.7). In this calculation the orthogonal splittings of $\nabla_a X_{bc}$, $\nabla_a Y_{bc}$, $\nabla_a Z_{bc}$, $\nabla_a \varepsilon_{bcd}$ must be used (see appendix A for the explicit expressions). The result is

$$\mathfrak{J}_{efa} = -L_f n_e n_a + L_e n_f n_a + \tilde{J}_f n_a + n_f \bar{J}_{ea} - n_e \bar{J}_{fa} + j_{efa}, \quad (4.10)$$

where

$$\tilde{J}_{ef} \equiv 2(X_{[f}^a + Y_{[f}^a)(\sigma_{e]a} + \omega_{e]a}) - 2(X_a^a + Y_a^a)\omega_{ef} + 2\varepsilon_{ab[e} D_{f]} Z^{ab}, \quad (4.11)$$

$$\begin{aligned} \bar{J}_{ea} \equiv & 2Y_e^b \sigma_{ab} + X_b^b \sigma_{ae} + h_{ae} \left(-\frac{1}{3} X_b^b \theta + X^{bc} \sigma_{bc} \right) + (-2X_a^b + Y_a^b) \sigma_{eb} + \\ & + 2Y_e^b \omega_{ab} - (X_b^b + 2Y_b^b) \omega_{ae} - Y_a^b \omega_{eb} + \varepsilon_{ebc} (-A^b Z_a^c - D_a Z^{bc} + D^c Z_a^b) + \\ & + \varepsilon_{abc} (-A^b Z_e^c + D_e Z^{bc}) + \frac{1}{3} (X_{ae} \theta - 3(\mathcal{L}_n Y_{ae})), \end{aligned} \quad (4.12)$$

$$L_e \equiv A_e (X_a^a + Y_a^a) - A^a (X_{ea} + Y_{ea}) + 2\omega^a Z_{[ae]} + \varepsilon_{abc} Z^{ab} \sigma_e^c + D_e Y_a^a, \quad (4.13)$$

$$\begin{aligned} j_{efa} \equiv & 2\omega_f Z_{[ea]} + 2\omega_e Z_{[af]} - 6\omega^b (h_{af} Z_{[eb]} + h_{ae} Z_{[bf]}) + 4\omega_a Z_{[ef]} + \\ & 2\varepsilon_{bc[f} Z^{bc} \left(\frac{1}{3} h_{e]a} \theta + \sigma_{e]a} \right) + 2h_{a[f} D_{e]} X_b^b + 2D_{[f} X_{e]a} + 2D_{[f} Y_{e]a}. \end{aligned} \quad (4.14)$$

From these expressions we deduce the properties $\tilde{J}_{[ab]} = \tilde{J}_{ab}$, $j_{[ab]c} = j_{abc}$.

4.3. The Bel and Bel-Robinson tensors

Finding a gravitational equivalent of the electromagnetic energy-momentum tensor T_{ab} proves to be a delicate issue. The reason for this lies in the impossibility of a local definition of the *gravitational energy-momentum density* due to the equivalence principle. Therefore it is clear from the very beginning that any tensor qualifying as the gravitational counterpart of T_{ab} must represent a physical quantity different from “energy-momentum”. If we are unwilling to introduce “new quantities” in physics then the point of view traditionally adopted consists in resorting to quantities defined non locally or using *pseudo-tensors* (a very good review of the research carried out in this direction is [41]). However, if we are ready to deal with a quantity different from “energy-momentum” then we find that it is possible to construct a tensor whose mathematical properties are similar to the electromagnetic tensor T_{ab} and this is the Bel tensor.

The Bel tensor was first introduced in [3] in connection with the construction of covariant divergences of quantities quadratic in the Riemann tensor. The original definition given by Bel can be shortened to the expression

$$B_a{}^b{}_c{}^d \equiv \frac{1}{2}(R_{apcq}^* R^{*bpdq} + {}^*R_{apcq} {}^*R^{bpdq} + {}^*R_{apcq}^* R^{*bpdq} + R_{apcq} R^{bpdq}), \quad (4.15)$$

which is formally similar to the first equation in (3.2) although with more terms due to the fact that the Riemann tensor has two blocks of antisymmetric indices.

If we expand the duals in (4.15) we get

$$\begin{aligned} B_{abcd} = & R_{aecf} R_b{}^e{}_d{}^f - \frac{1}{2} g_{dc} R_{aefp} R_b{}^e{}_f{}^p - \frac{1}{2} g_{ba} R_{cefp} R_d{}^e{}_f{}^p + R_b{}^e{}_c{}^f R_{dfae} + \\ & + \frac{1}{8} g_{ba} g_{dc} R_{efph} R^{efph}. \end{aligned} \quad (4.16)$$

The Bel tensor has a number of remarkable mathematical properties which are summarized next.

Theorem 4.1 *The following statements hold true for the Bel tensor*

- (i) $B_{abcd} = B_{(ab)(cd)} = B_{cdab}$, $B^a{}_{acd} = 0$.
- (ii) (Generalized dominant property) If u_1^a , u_2^a , u_3^a , u_4^a are arbitrary causal, future directed vectors then $B_{abcd} u_1^a u_2^b u_3^c u_4^d \geq 0$.
- (iii) $B_{abcd} = 0 \iff R_{abcd} = 0 \iff \exists$ a timelike vector u^a such that $B_{abcd} u^a u^b u^c u^d = 0$.
- (iv) Equation (4.1) entails

$$\nabla_a B^a{}_{bcd} = \mathfrak{J}_d{}^{ae} R_{beca} + \mathfrak{J}_c{}^{ae} R_{beda} - \frac{1}{2} g_{cd} \mathfrak{J}^{aef} R_{bfae}, \quad \nabla_a \mathfrak{J}_{bc}{}^a = 0. \quad (4.17)$$

The similarity between the mathematical properties of B_{abcd} presented in this theorem and those of T_{ab} given by theorem 3.1 is apparent. Therefore the Bel tensor fulfills the basic mathematical requirements needed for it to be regarded as the gravitational counterpart of the energy-momentum tensor in electromagnetism. In vacuum, the Bel tensor acquires a simpler expression which is

$$T_{abcd} \equiv C_a{}^p{}_d{}^f C_{bpcf} + C_a{}^p{}_c{}^f C_{bpdf} - \frac{1}{8} g_{ab} g_{cd} C_{pqrs} C^{pqrs}, \quad (4.18)$$

where $R_{abcd} = C_{abcd}$ has been used. The tensor T_{abcd} is known as the Bel-Robinson tensor [2] and it can be defined in any spacetime, whether vacuum or not, by means

of equation (4.18). All the properties of theorem 4.1 except point (iv) are also true for the Bel-Robinson tensor with the following changes: T_{abcd} is totally symmetric and trace-free and in point (iii) the Riemann tensor must be replaced by the Weyl tensor. If $R_{ab} = \Lambda g_{ab}$ (Einstein space) then the covariant divergence of the Bel-Robinson tensor takes a particularly simple form.

$$\nabla_a T^a_{bcd} = 0.$$

A full account of the properties reviewed here of the Bel and Bel-Robinson tensors together with their proofs can be found in [38] and [10]. In the former reference a generalization of (3.2) and (4.15) valid for any tensor is put forward. Tensors resulting from this generalization are called *superenergy tensors* and they all fulfill the generalized dominant property (*generalized dominant superenergy condition*).

What about the physical role of the Bel tensor? This question has been addressed many times in the past and no definitive answer exists. Bel himself proposed the name of *superenergy* for the physical quantity which might lie behind the Bel tensor (this physical quantity would be represented by the components of Bel tensor in a suitable frame). If we denote by L the basic unit in the geometrized system then from the definition of the Bel tensor we deduce that the physical units of superenergy are L^{-4} which can be interpreted as either energy density squared or energy density per unit area. Both interpretations have been researched in the literature and the opinion favoring the second interpretation seems to have gained weight. For a history of the different interpretations of the Bel tensor which have been studied in the past see [38] and references therein.

In the case of electromagnetism we have seen that a full understanding of the physical properties of the electromagnetic energy-momentum tensor can be achieved by the Poynting theorem. This theorem is nothing but the orthogonal splitting of (3.3) and the different equations of this splitting inform us of the evolution of the different parts of the electromagnetic energy-momentum tensor. Therefore it is expected that the orthogonal splitting of equation (4.17) will yield valuable information about the true physical role of the Bel tensor. The calculation of such an orthogonal splitting is accomplished in the forthcoming sections.

5. Orthogonal splitting of the Bel-Robinson tensor

Before studying the general case of the Bel tensor we calculate the orthogonal splitting of the Bel-Robinson tensor. The different parts of the splitting take simpler forms and they will give us valuable insights about the general case. To calculate this splitting we insert the expression for the orthogonal splitting of the Weyl tensor given by (4.9) into (4.18). After some computations we get

$$T_{abcd} = W n_a n_b n_c n_d + 4\mathcal{P}_{(a} n_b n_c n_{d)} + 6t_{(ab} n_c n_{d)} + 4Q_{(abc} n_{d)} + t_{abcd}, \quad (5.1)$$

where

$$\begin{aligned} W &\equiv E_{ab}E^{ab} + B_{ab}B^{ab}, \quad \mathcal{P}_a \equiv 2B_p{}^l E_{ql} \varepsilon_a{}^{pq}, \quad t_{ab} \equiv W h_{ab} - 2(B_a{}^c B_{bc} + E_a{}^c E_{bc}), \\ Q_{cdb} &\equiv h_{cd} \mathcal{P}_b - 2(B_{da} E_{cf} + B_{ca} E_{df}) \varepsilon_b{}^{af}, \\ t_{abcd} &\equiv 4(B_{ab} B_{cd} + E_{ab} E_{cd}) - h_{cd} t_{ab} + 2h_{b(d} t_{c)a} + 2h_{a(d} t_{c)b} - h_{ab} t_{cd} + \\ &\quad + W(h_{ab} h_{cd} - 2h_{a(c} h_{d)b}). \end{aligned} \quad (5.2)$$

Some of these quantities have been obtained before and have found diverse applications. The scalar W (superenergy density) and the spatial vector \mathcal{P}^a , called the super-Poynting vector, were first used in [5] to define intrinsic *radiation states* in gravitation theory (see section 8 for more details about this) and the tensor t_{ab} was used in [11] to show the causal propagation of gravity in vacuum (also the role of t_{ab} in the definition of radiation states was discussed in this reference). We establish next the basic algebraic properties of these quantities

Proposition 5.1 *The following basic algebraic properties hold*

- (i) $t_{(ab)} = t_{ab}$, $Q_{(abc)} = Q_{abc}$, $t_{(abcd)} = t_{abcd}$,
- (ii) $t^a_a = W \geq 0$, $Q^a_{ab} = \mathcal{P}_b$, $t^a_{abc} = t_{bc}$,
- (iii) Q_{abc} and t_{abcd} contain all the information about the Bel-Robinson tensor.

Proof : Points (i) and (ii) can be proven directly from the tensor expressions given in (5.2) but it is far more easier to use (5.1) and write each part of the decomposition in terms of the Bel-Robinson tensor T_{abcd} . The result is

$$W = T_{abcd}n^a n^b n^c n^d, \quad \mathcal{P}_e = -T_{ebcd}h^e_a n^b n^c n^d, \quad t_{ab} = T_{pqrs}h^p_a h^q_b n^r n^s, \quad (5.3)$$

$$Q_{abc} = -T_{pqrs}h^p_a h^q_b h^r_c n^s, \quad t_{abcd} = T_{pqrs}h^p_a h^q_b h^r_c h^s_d. \quad (5.4)$$

The symmetries expressed in point (i) are now a consequence of the total symmetry of T_{abcd} . Point (ii) is straightforward either from (5.2) or from (5.3)-(5.4) and the complete tracelessness of the Bel-Robinson tensor. Thus given t_{abcd} and Q_{abc} it is evident from their algebraic properties that we recover the remaining parts of the orthogonal decomposition of the Bel-Robinson tensor which proves point (iii). \square

Remark 5.1 We can obtain an independent proof of point (iii) of the previous proposition if we count the number of independent components of Q_{abc} and t_{abcd} and compare their sum with the number of total independent components of T_{abcd} . The respective numbers are

$$\begin{aligned} \text{number of independent components of } T_{abcd} &= 25, \\ \text{number of independent components of } t_{abcd} &= 15, \\ \text{number of independent components of } Q_{abc} &= 10, \\ 10 + 15 &= 25. \end{aligned}$$

Proposition 5.2

$$t_{abcd} = 0 \iff t_{ab} = 0 \iff W = 0 \iff C_{abcd} = 0,$$

Proof : From proposition 5.1 we deduce $t_{abcd} = 0 \implies t_{ab} = 0 \implies W = 0$. To prove the converse, let us assume that $W = 0$. In this case the first equation of (5.3) implies

$$T_{abcd}n^a n^b n^c n^d = 0.$$

Combining this with point (iii) of theorem 4.1 applied to the Bel-Robinson tensor we deduce $C_{abcd} = 0$. Trivially, $C_{abcd} = 0$ implies $T_{abcd} = 0$ and thus t_{ab} , t_{abcd} vanish as well. \square

The importance of this result lies in the fact that evaluation of any of the quantities W , t_{ab} , t_{abcd} enables a observer represented by the unit timelike vector n^a to decide if the purely gravitational part of the Riemann tensor (or the Riemann tensor itself if we are in a vacuum spacetime) is present or not. Also the variation of these quantities along the integral curves of n^a should give a measure of how the Weyl tensor changes for this observer. We will turn back to this important point in section 7.

5.1. Canonical forms for the different Petrov types

We can obtain more interesting properties of the quantities introduced in (5.2) if we set up a suitable orthonormal frame. Such a frame arises in the calculation of the canonical forms which E_{ab} and B_{ab} take for the different *Petrov types*. These canonical forms are reviewed in appendix B and we refer the reader to this appendix for more details. The results presented in this subsection are algebraic in nature and should be understood as formulated in the tangent space of a point.

Proposition 5.3 *The tensor Q_{abc} vanishes if and only if E_{ab} , B_{ab} are linearly dependent.*

Proof : From (5.2) it is easy to show that Q_{abc} is zero if E_{ab} and B_{ab} are linearly dependent. Now, if $Q_{abc} = 0$ then from point (ii) of proposition 5.1 we get $\mathcal{P}^a = 0$. This last condition can be re-written in the form

$$E_a{}^r B_{rb} - E_b{}^r B_{ra} = 0, \quad (5.5)$$

from which we conclude that the endomorphisms represented by $E^a{}_b$, $B^a{}_b$ commute. This is only possible for Petrov types I and D as can be easily checked using the canonical forms of appendix B (alternatively, two symmetric endomorphisms have a common basis of eigenvectors if and only if they commute). For Petrov type D trivially E_{ab} and B_{ab} are linearly dependent, so we will assume that the spacetime is of Petrov type I. In the orthonormal frame of (B.1) we find that the only nonvanishing component of Q_{abc} is

$$Q_{123} = -2(B_{11}E_{22} - B_{22}E_{11}),$$

and hence $Q_{123} = 0$ implies $B_{11}E_{22} = B_{22}E_{11}$ from which we deduce from (B.1) that E_{ab} and B_{ab} are linearly dependent (recall that $E_{11} + E_{22} + E_{33} = B_{11} + B_{22} + B_{33} = 0$). \square

From this result we deduce that Q_{abc} resembles in its mathematical properties the electromagnetic Poynting vector. We will see later that if we are to study the radiation of superenergy then Q_{abc} (or any equivalent tensor thereof) will take over the role of the Poynting vector.

Proposition 5.3 admits the following corollary.

Corollary 5.1

- (i) $Q_{abc} \neq 0 \implies$ Petrov type is either II, III, N or I.
- (ii) If Petrov type is II, III, or N $\implies Q_{abc} \neq 0$.
- (iii) Petrov type D is the only type in which Q_{abc} always vanishes.

Proposition 5.4 *The following algebraic properties hold*

- (i) For Petrov type III we have

$$t_{ab} = \frac{1}{2}h_{ab}W - \frac{2\mathcal{P}_a\mathcal{P}_b}{W}, \quad Q_{abc} = 3h_{(bc}\mathcal{P}_{a)} - \frac{16\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{W^2},$$

$$t_{abcd} = -\frac{64\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c\mathcal{P}_d}{W^3} + \frac{12}{W}h_{(ab}\mathcal{P}_{c)}\mathcal{P}_d, \quad \mathcal{P}_a \neq 0, \quad \mathcal{P}_a\mathcal{P}^a = \frac{W^2}{4}.$$

The two independent principal null directions of the Weyl tensor (see e.g. [40]) can be calculated explicitly yielding

$$k_1^a = -\mathcal{P}^a + \frac{1}{2}n^aW, \quad k_2^a = \mathcal{P}^a + \frac{1}{2}n^aW. \quad (5.6)$$

(ii) For Petrov type N we have

$$t_{ab} = \frac{\mathcal{P}_a \mathcal{P}_b}{W}, \quad Q_{abc} = \frac{\mathcal{P}_a \mathcal{P}_b \mathcal{P}_c}{W^2}, \quad t_{abcd} = \frac{\mathcal{P}_a \mathcal{P}_b \mathcal{P}_c \mathcal{P}_d}{W^3}, \quad \mathcal{P}_a \neq 0, \quad \mathcal{P}_a \mathcal{P}^a = W^2.$$

In this case the only independent principal null direction of the Weyl tensor is

$$k^a \equiv W n^a + \mathcal{P}^a. \quad (5.7)$$

Proof : The proof of this result consists in using the canonical forms for Petrov types III and N written in appendix B to find canonical forms for t_{ab} , \mathcal{P}^a , Q_{abc} and t_{abcd} . These canonical forms lead then to the expressions presented in points (i) and (ii). We detail next this procedure for each of the Petrov types.

– *Petrov type III:* using the frame of (B.4) we get

$$-Q_{133} = -Q_{122} = Q_{111} = 2(B_{12}^2 + E_{12}^2), \quad \mathcal{P}_1 = -2(E_{12}^2 + B_{12}^2), \quad t_{22} = t_{33} = 2(E_{12}^2 + B_{12}^2).$$

with all the other components of Q_{abc} , \mathcal{P}^a , t_{ab} being zero. From these expressions we deduce

$$Q_{abc} = h_{bc}\mathcal{P}_a + h_{ac}\mathcal{P}_b + h_{ab}\mathcal{P}_c - \frac{\mathcal{P}_a \mathcal{P}_b \mathcal{P}_c}{B_{12}^2 + E_{12}^2}, \quad t_{ab} = 2(E_{12}^2 + B_{12}^2)h_{ab} - \frac{\mathcal{P}_a \mathcal{P}_b}{2(E_{12}^2 + B_{12}^2)}, \quad (5.8)$$

and using point (ii) of proposition 5.1 we conclude

$$B_{12}^2 + E_{12}^2 = \frac{\sqrt{\mathcal{P}_a \mathcal{P}^a}}{2}, \quad \mathcal{P}_a \mathcal{P}^a = \frac{W^2}{4}.$$

Replacing this back in (5.8) we obtain the expressions sought for t_{ab} and Q_{abc} . Inserting the values just found for t_{ab} in the formula for t_{abcd} of (5.2) yields

$$t_{abcd} = 4(B_{ab}B_{cd} + E_{ab}E_{cd}) + \frac{2h_{ab}\mathcal{P}_c\mathcal{P}_d}{W} - \frac{4\mathcal{P}_b}{W}h_{a(d}\mathcal{P}_{c)} + \frac{2\mathcal{P}_a}{W}(h_{cd}\mathcal{P}_b - h_{bd}\mathcal{P}_c - h_{bc}\mathcal{P}_d).$$

Again using the canonical forms of (B.4) we transform the term $4(B_{ab}B_{cd} + E_{ab}E_{cd})$ into

$$-\frac{64\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c\mathcal{P}_d}{W^3} + \frac{4}{W}(\mathcal{P}_c(h_{bd}\mathcal{P}_a + h_{ad}\mathcal{P}_b) + (h_{bc}\mathcal{P}_a + h_{ac}\mathcal{P}_b)\mathcal{P}_d).$$

Combining the last two equations we find the expression for t_{abcd} given in the proposition. It is now a simple calculation to check that the vectors k_1^a and k_2^a are indeed null and that they fulfill the properties

$$T_{abcd}k_1^ak_1^bk_1^ck_1^d = 0, \quad T_{abcd}k_2^ak_2^bk_2^ck_2^d = 0$$

which implies that k_1^a and k_2^a are the Weyl tensor principal null directions (see [36] p. 328).

– *Petrov type N:* In this case, we obtain in the frame of (B.5)

$$Q_{111} = \mathcal{P}_1 = -t_{11} = -4(B_{22}^2 + E_{22}^2),$$

with the other components vanishing. Hence

$$Q_{abc} = \frac{\mathcal{P}_a\mathcal{P}_b\mathcal{P}_c}{16(B_{22}^2 + E_{22}^2)^2}, \quad t_{cd} = \frac{\mathcal{P}_c\mathcal{P}_d}{4(B_{22}^2 + E_{22}^2)} \Rightarrow B_{22}^2 + E_{22}^2 = \frac{\sqrt{\mathcal{P}_a\mathcal{P}^a}}{4} = \frac{W}{4}.$$

Similarly, working in the canonical frame we obtain that the only nonvanishing component of t_{abcd} is

$$t_{1111} = 4(B_{22}^2 + E_{22}^2).$$

Combining the previous pair of equations the expressions of point (ii) follow. Also it is a simple matter to check that k^a is null and that $T_{abcd}k^ak^bk^ck^d = 0$. \square

An important result of this proposition is that for Petrov types III and N the Bel-Robinson tensor is characterized by just two independent quantities which are W and \mathcal{P}^a and thus we can say that the number of algebraically independent components of the Bel-Robinson tensor is two for these Petrov types. This is not true of the other Petrov types and therefore some conclusions drawn from considerations involving type III and N might not carry over to other Petrov types. An example of this is the definition and study of gravitational radiation using the Bel-Robinson tensor where, traditionally, a nonvanishing vector \mathcal{P}^a for any observer n^a has been regarded as an *intrinsic state* of gravitational radiation [5] (see definition 8.1). We will see in section 7 that this condition is not general enough and indeed in certain Petrov type I spacetimes we can still speak of an intrinsic state of gravitational radiation with \mathcal{P}^a being zero.

6. Orthogonal splitting of the Bel tensor

The orthogonal splitting of the Bel tensor is obtained by replacing the expression for the Riemann tensor given by (4.6) in (4.16) with the result

$$\begin{aligned} B_{abcd} = & \overline{W}n_an_bn_cn_d + 4\overline{\mathcal{P}}_{(a}n_bn_cn_d) + 2n_{(a}\overline{\mathcal{Q}}_{b)cd} + 2n_{(d}\overline{\mathcal{Q}}_{c)ab} + \\ & + \overline{t}_{ab}n_cn_d + \overline{t}_{cd}n_an_b + 4n_{(a}t_{b)(c}^*n_d) + \overline{t}_{abcd}. \end{aligned} \quad (6.1)$$

Each of the spatial parts of the Bel tensor is defined as follows

$$\begin{aligned} \overline{W} & \equiv \frac{1}{2}(X_{ab}X^{ab} + Y_{ab}Y^{ab}) + Z_{ab}Z^{ab}, \quad \overline{\mathcal{P}}_a \equiv \varepsilon_{abc}(Y_d{}^cZ^{bd} - X_d{}^cZ^{db}), \\ \overline{t}_{cd} & \equiv h_{cd}\overline{W} - X_c{}^aX_{da} - Y_c{}^aY_{da} - Z_{ac}Z_d{}^a - Z_c{}^aZ_{da}, \\ t_{bd}^* & \equiv 2X_{(d}{}^aY_{b)a} - X_{bd}Y_a{}^a - Y_{bd}X_a{}^a + h_{bd}(-X^{ac}Y_{ac} + Z^{ac}Z_{ac} + X_a{}^aY^c{}_c) - \\ & \quad - Z_b{}^aZ_{ad} - Z_b{}^aZ_{da} \\ \overline{\mathcal{Q}}_{bcd} & \equiv h_{cd}\overline{\mathcal{P}}_b + 2Z_{(d}{}^a(-Y_{c)}{}^e\varepsilon_{bae} + \varepsilon_{c)ba}X_e{}^e + \varepsilon_{c)ae}X_b{}^e) + \\ & \quad + 2Z^{ae}(h_{b(c}\varepsilon_{d)ae}X_f{}^f - \varepsilon_{d)af}X_e{}^f) - X_{e(d}\varepsilon_{c)ba} + h_{cd}X_a{}^f\varepsilon_{bef} - X_{b(d}\varepsilon_{c)ae}). \end{aligned}$$

The expression for \overline{t}_{abcd} is a bit long and is omitted (its explicit form is not needed in this paper).

Proposition 6.1 *The tensors \overline{t}_{ab} , t_{ab}^* , $\overline{\mathcal{Q}}_{abc}$ and \overline{t}_{abcd} satisfy the following basic algebraic properties*

$$\begin{aligned} \overline{t}_{(ab)} & = \overline{t}_{ab}, \quad t_{(ab)}^* = t_{ab}^*, \quad \overline{\mathcal{Q}}_{a(bc)} = \overline{\mathcal{Q}}_{abc}, \quad \overline{t}_{(ab)cd} = \overline{t}_{abcd} = \overline{t}_{cdab}, \quad \overline{t}^a{}_a = \overline{W}, \quad \overline{\mathcal{Q}}_b{}^a{}_a = \overline{\mathcal{P}}_b, \\ \overline{t}^a{}_{abc} & = \overline{t}_{bc} \end{aligned}$$

Proof : These properties can be proven from a direct computation using the definitions of \overline{t}_{ab} , t_{ab}^* , $\overline{\mathcal{Q}}_{abc}$ and t_{abcd} given above but this results in involved

calculations even when done by computer. A simpler procedure is to start with (6.1) and derive the relations

$$\overline{W} = B_{abcd}n^a n^b n^c n^d, \quad \overline{\mathcal{P}}_a = -B_{pbcd}h^p_a n^b n^c n^d, \quad (6.2)$$

$$\overline{t}_{ab} = B_{pqcd}n^c n^d h^p_a h^q_b, \quad t^*_{ab} = B_{rpsq}n^p n^q h^r_a h^s_b, \quad \overline{Q}_{abc} = -B_{pqrs}n^p h^q_a h^r_b h^s_c, \quad (6.3)$$

$$\overline{t}_{abcd} = B_{pqrs}h^p_a h^q_b h^r_c h^s_d. \quad (6.4)$$

From these relations and the properties of the Bel tensor it is straightforward to prove the proposition. \square

Remark 6.1 An important consequence of the algebraic properties presented in this last result is that \overline{t}_{abcd} , \overline{Q}_{abc} and t^*_{ab} contain all the information about the Bel tensor. As we did in the case of the Bel-Robinson tensor we can count the number of independent components of these tensors and check that they add up to the number of independent components of the Bel tensor

$$\begin{aligned} &\text{number of independent components of } B_{abcd} = 45, \\ &\text{number of independent components of } \overline{t}_{abcd} = 21, \\ &\text{number of independent components of } \overline{Q}_{abc} = 18, \\ &\text{number of independent components of } t^*_{ab} = 6. \\ &21 + 18 + 6 = 45. \end{aligned}$$

Proposition 6.2

$$\overline{W} = 0 \iff \overline{t}_{ab} = 0 \iff \overline{t}_{abcd} = 0 \iff R_{abcd} = 0, \quad (\text{no superenergy} \iff \text{no gravitation})$$

Proof : If R_{abcd} vanishes then so does B_{abcd} and trivially $\overline{W} = 0$, $\overline{t}_{ab} = 0$, $\overline{t}_{abcd} = 0$. Assume now that \overline{W} is zero. In that case point (iii) of theorem 4.1 entails $R_{abcd} = 0$ thus proving the desired result. \square

We finish this section by pointing out that whenever the Bel and Bel-Robinson tensors are equal then we deduce the relations

$$t^*_{ab} = \overline{t}_{ab} = t_{ab}, \quad \overline{Q}_{abc} = Q_{abc}, \quad \overline{t}_{abcd} = t_{abcd},$$

from which we conclude that $\overline{W} = W$, $\overline{\mathcal{P}}_a = \mathcal{P}_a$. The Bel and the Bel-Robinson tensors are equal if and only if $R_{ab} = 0$ (see Corollary 6.1 of [38]).

7. Dynamical laws of superenergy

In this section we present the most important result of this paper which is the orthogonal splitting of (4.17). As explained before this result is analogous to (3.7)-(3.9) and this analogy will enable us to extract some interesting conclusions as to the interpretation of certain parts of the orthogonal splitting of the Bel tensor.

Before presenting the results we should make some remarks concerning the calculations. In order to work out the orthogonal splitting of (4.17) neither (4.1), nor its orthogonal splitting is needed. This is similar to electromagnetism, where the Maxwell equations are not needed to obtain (3.7)-(3.9). The orthogonal splitting of (4.17) is calculated by inserting the orthogonal splitting of each of the quantities appearing in this equation (the Bel tensor, the Riemann tensor and the matter current) and then using the orthogonal splitting of the different terms which appear in the resulting expressions. Here we only provide the final expressions referring the reader to appendix A for more details about the intermediate steps in the calculations.

Theorem 7.1 (Dynamical laws of superenergy) *The equation*

$$\nabla_a B^a_{bcd} = \mathfrak{J}_d^{ae} R_{beca} + \mathfrak{J}_c^{ae} R_{beda} - \frac{1}{2} g_{cd} \mathfrak{J}^{aef} R_{bf ae}$$

is equivalent to the following set of expressions

$$\begin{aligned} \mathcal{L}_n \bar{\mathcal{P}}_c + D_a t^{*a}_c + \left(\bar{\mathcal{P}}_c + \frac{2\bar{\mathcal{Q}}_{ca}}{3} \right) \theta - j_c^{ae} Y_{ae} + 2\sigma_{ae} \bar{\mathcal{Q}}^a_c{}^e - Z^f_e \varepsilon_{caf} \bar{\mathcal{J}}^{ae} + \\ + 2\omega^{eb} (\bar{\mathcal{Q}}_{ecb} - h_{ce} \bar{\mathcal{P}}_b) + A^a (h_{ca} \bar{W} + \bar{t}_{ca} + 2t_{ca}^*) = 0, \end{aligned} \quad (7.1)$$

$$\begin{aligned} D_a t^{*a}_c - D_a \bar{t}^a_c = L^a Y_{ac} + 2Z^e_{[a} \varepsilon_{b]ce} \bar{\mathcal{J}}^{ab} - \frac{1}{2} Z^a_c \varepsilon_{abe} \bar{J}^{be} + \\ + j^{abe} (h_{ac} (-X^d_d h_{be} + X_{be} + Y_{be}) - h_{ae} X_{bc}), \end{aligned} \quad (7.2)$$

$$\begin{aligned} \mathcal{L}_n \bar{t}_{cd} + D_a \bar{\mathcal{Q}}^a_{cd} + \omega^{af} \Omega_{afcd} + \sigma^{af} \Sigma_{afcd} + \frac{2}{3} (\bar{t}_{cd} + t_{cd}^*) \theta + 2A^a (h_{a(d} \bar{\mathcal{P}}_{c)}) + \bar{\mathcal{Q}}_{acd}) - \\ - j^{fhd} \left(\frac{1}{2} \varepsilon_{afh} h_{cd} + 2h_{f(d} \varepsilon_{c)ah} \right) Z^a_b + (h_{cd} Y_{ae} - h_{da} Y_{ce} - h_{ca} Y_{de}) \bar{\mathcal{J}}^{ae} = 0, \end{aligned} \quad (7.3)$$

$$\begin{aligned} \mathcal{L}_n t_{bd}^* + D_a \bar{\mathcal{Q}}^a_{bd} + \omega^{af} \Omega_{afbd}^* + \sigma^{af} \Sigma_{afbd}^* + \frac{1}{3} (\bar{t}_{bd} + 2t_{bd}^* + \bar{t}_b{}^a{}_{da}) \theta + \\ A^a \left(2h_{a(d} \bar{\mathcal{P}}_{b)} + \bar{\mathcal{Q}}_{abd} + \bar{\mathcal{Q}}_{(bd)a} \right) - j^{cfh} \varepsilon_{ha(d} h_{b)f} Z^a_c - L^a \varepsilon_{ae(d} Z^e_{b)} + \tilde{J}^{ae} h_{a(d} Y_{b)e} + \\ + (h_{a(d} X_{b)e} + h_{e(d} (X_{b)a} - h_{b)a} X^p_p) - h_{bd} (X_{ae} - h_{ae} X^p_p) - h_{ae} X_{bd}) \bar{\mathcal{J}}^{ae} = 0, \end{aligned} \quad (7.4)$$

$$\begin{aligned} \mathcal{L}_n \bar{\mathcal{Q}}_{bcd} + D_a \bar{t}^a_{bcd} + \omega^{af} \Pi_{afbcd} + \sigma^{af} \Delta_{afbcd} + \theta \bar{\mathcal{Q}}_{(bcd)} + A^a (\bar{t}_{abcd} + h_{ab} \bar{t}_{cd} + 2h_{a(d} t_{c)b}^*) + \\ + \tilde{J}^{ef} Z^a_b \left(\frac{1}{2} \varepsilon_{aef} h_{cd} + 2h_{e(d} \varepsilon_{c)af} \right) - \bar{\mathcal{J}}^{ae} \varepsilon_{bef} (h_{cd} Z^f_a - 2h_{a(d} Z^f_{c)}) + \\ + L^a (h_{cd} Y_{ab} - 2h_{a(d} Y_{c)b}) + j^{aef} H_{aefbcd} = 0, \end{aligned} \quad (7.5)$$

where

$$\begin{aligned} \Sigma_{afcd} &\equiv -2h_{a(d} \bar{t}_{c)f} + 2h_{a(d} t_{c)f}^* + \bar{t}_{cdf}, \quad \Omega_{afcd} \equiv -2h_{a(d} \bar{t}_{c)f} - 2h_{a(d} t_{c)f}^*, \\ \Sigma_{afbd}^* &\equiv \bar{t}_{abdf} + h_{a(d} (\bar{t}_{b)f} - t_{b)f}^*), \quad \Omega_{afbd}^* \equiv -h_{a(d} (\bar{t}_{b)f} + 3t_{b)f}^*), \\ \Pi_{afbcd} &\equiv -2\bar{\mathcal{Q}}_{fcd} h_{ab} - 4\bar{\mathcal{Q}}_{(bf)(c} h_{d)a}, \quad \Delta_{afbcd} \equiv 4\bar{\mathcal{Q}}_{[fb](c} h_{d)a}, \\ H_{aefbcd} &\equiv 2h_{ab} h_{e(d} X_{c)f} + 2h_{ef} h_{a(d} X_{c)b} + 2h_{ad} h_{c[b} X_{f]e} + 4h_{c[d} h_{a][f} X_{b]e} + \\ &\quad + X^h_h (h_{ab} h_{cd} h_{ef} - 2h_{ab} h_{c(f} h_{e)d} - 2h_{ef} h_{a(d} h_{c)b}) \end{aligned} \quad (7.6)$$

Proof : See appendix A. □

Theorem 7.2 (Matter current conservation) *The equation $\nabla_a \mathfrak{J}_{bc}^a = 0$ is equivalent to the expressions*

$$\begin{aligned} \mathcal{L}_n L_p = A^q (\tilde{J}_{pq} - \bar{J}_{pq}) + \frac{\theta}{3} (j_p{}^q{}_q - 2L_p) + (L_b h_{pa} - j_{apb}) \sigma^{ab} + (j_{apb} + L_b h_{pa}) \omega^{ab} - \\ - D_q \bar{J}_p{}^q, \end{aligned} \quad (7.7)$$

$$\begin{aligned} \mathcal{L}_n \tilde{J}_{bp} = D_q j_{bp}{}^q + \frac{1}{3} (2\bar{J}_{[bp]} - \tilde{J}_{bp}) \theta + A^q (2h_{q[p} L_{b]} - j_{bpq}) + \\ + 2\sigma^{ac} (h_{a[p} \bar{J}_{b]c} + h_{a[p} \tilde{J}_{b]c}) + 2\omega^{ac} (-h_{a[p} \bar{J}_{b]c} + h_{a[p} \tilde{J}_{b]c}). \end{aligned} \quad (7.8)$$

Proof : Again see appendix A. \square

Remark 7.1 Equations (7.1)-(7.5) and (7.7)-(7.8) can be regarded as the gravitational counterpart of (3.7)-(3.9). They form an *inhomogeneous* evolution system for the variables $\bar{\mathcal{P}}_a$, \bar{t}_{ab} , t_{ab}^* , \bar{Q}_{abc} , L_a and \bar{J}_{ab} . The inhomogeneous part (source) of each equation consists of those terms which contain neither kinematical quantities nor spatial covariant derivatives. These terms play the same role as $-E_a J^a$ in (3.7) (power lost by the charge flux) and $\varepsilon_{acb} B^b J^c + E_a \rho$ (change of momentum due to charges) in (3.8). We also find that no expressions for $\mathcal{L}_n t_{abcd}$, $\mathcal{L}_n \bar{J}_{ab}$, $\mathcal{L}_n j_{abc}$ are supplied by the orthogonal splitting of (4.17) and in fact only by using the full content of (4.1) can such expressions be found.

Remark 7.2 The evolution equations of theorems 7.1 and 7.2 are written in such a way that the coupling of the kinematical quantities to the different parts of the orthogonal decomposition of the Bel tensor and the matter current is manifest. Note also that in these equations we can find terms which do not contain kinematical quantities. As the kinematical quantities can be always set to zero at a given point by choosing a suitable vector field n^a we deduce that any term containing explicitly a kinematic quantity is *observer dependent* and it will play a similar role as the inertial terms in equations (3.7)-(3.9) found for electromagnetism.

Taking the trace of (7.3) we find

$$\mathcal{L}_n \bar{W} + D_a \bar{\mathcal{P}}^a + \sigma^{ae} (\bar{t}_{ae} + 2t_{ae}^*) + \frac{2\theta}{3} (t^{*a}{}_a + 2\bar{W}) + \frac{1}{2} \varepsilon_{afb} j^{fb}{}_e Z^{ae} + Y^{ae} \bar{J}_{ae} + 4A^a \bar{\mathcal{P}}_a = 0. \quad (7.9)$$

Equations similar to this one have been used in different places of the literature principally with the aim of controlling the evolution of the scalar \bar{W} [1, 27].

7.1. Dynamical laws of superenergy in vacuum

Theorem 7.1 assumes a far more simpler form in vacuum because the covariant divergence of the Bel tensor takes the simpler form $\nabla_a T^a{}_{bcd} = 0$. The specific result in this case is given in the next theorem.

Theorem 7.3 *The equation*

$$\nabla_a T^a{}_{bcd} = 0,$$

is equivalent to the following set of expressions

$$\mathcal{L}_n t_{cd} = -2A^a (h_{a(d} \mathcal{P}_{c)}) + Q_{cda} + 4\omega^{ab} h_{a(d} t_{c)b} - \frac{4}{3} t_{cd} \theta - t_{cdae} \sigma^{ae} - D_a Q^a{}_{cd}, \quad (7.10)$$

$$\mathcal{L}_n Q_{bcd} = -A^a (t_{bcd a} + 3h_{a(d} t_{bc)}) + 6\omega^{ae} h_{a(d} Q_{bc)e} - \theta Q_{bcd} - D_a t^a{}_{bcd}. \quad (7.11)$$

Proof : This can be regarded as a particular case of theorem 7.1 with $\mathfrak{J}_{abc} = 0$ and $B_{abcd} = T_{abcd}$. This entails $t_{ab} = t_{ab}^*$, $\bar{Q}_{abc} = Q_{abc}$, $\bar{t}_{abcd} = t_{abcd}$, $L_a = 0$, $\bar{J}_{ab} = 0$, $\bar{J}_{ab} = 0$, $j_{abc} = 0$ which used in (7.3) and (7.5) leads to (7.10) and (7.11). Equation (7.2) becomes an identity and (7.1) is now obtained by taking the trace of (7.3). \square

In the particular case studied in theorem (7.3) we find that (7.9) and (7.1) acquire simpler expressions which are

$$\mathcal{L}_n W = -4A^a \mathcal{P}_a - 2W\theta - 3t^{ae} \sigma_{ae} - D_a \mathcal{P}^a, \quad (7.12)$$

$$\mathcal{L}_n \mathcal{P}_d = -A^a (3t_{da} + h_{da} W) - \frac{5\theta}{3} \mathcal{P}_d - 2Q_{dae} \sigma^{ae} + 2\mathcal{P}^a \omega_{da} - D_a t^a{}_d. \quad (7.13)$$

The linearized form of (7.12) was known to Bel [5] and in fact he took this equation as the starting point for a definition of a state of *intrinsic radiation* for the gravitational field in vacuum (see subsection 8.1 for further details). The general form of (7.12) was derived in [32]. It is interesting to note the formal analogy of (7.12)-(7.13) with (3.7)-(3.8) where W and \mathcal{P}^a take the role of the electromagnetic energy density and the Poynting vector respectively. Although (7.13) has, as far as we know, never been obtained in its complete form, the knowledge of (7.12), even in its linearized form, shown in equation (8.1), has been enough to construct the analogy just mentioned and a lot of work has been devoted to studying the behaviour of gravitational systems by studying the super-energy density and the super -Poynting vector in the system –see for example [25, 13, 43, 23]. The results obtained are very suggestive but we must note that (7.12)-(7.13) are not equivalent to (7.10)-(7.11) which in fact contain more information. Therefore, if we are to study gravitational radiation by means of techniques involving the study of the evolution of the different spatial parts of Bel-Robinson tensor then we should start with the general equations (7.10)-(7.11). This matter is addressed in section 8.

8. Application: superenergy radiative states of the gravitational field

In electromagnetism, we speak of *electromagnetic radiation* to mean that electromagnetic energy is traveling from one part of a system to another which in turn implies the existence of a flux of energy-momentum. By the Poynting theorem this flux is represented by the Poynting vector and thus whenever the Poynting vector is not zero at a point we say that electromagnetic radiation is going through that point. This statement is observer dependent because in order to define the Poynting vector an observer n^a is needed (see equation (3.6)). Therefore we may find for example, that the Poynting vector is zero for one observer whereas another observer measures a non-vanishing Poynting vector. However, there are configurations in which any observer will measure a non-vanishing Poynting vector and in these cases it is said that the electromagnetic field is in a radiation state at the point. From an algebraic point of view this can only happen if the electromagnetic field F_{ab} is *singular* or *null* which means that it can be written as the exterior product of a null and a spatial vector. (see e.g. [34]).

If we try to follow the same procedure to define gravitational radiation in General Relativity we are immediately confronted with the fact that, due to the equivalence principle, we can always find an observer who measures no “gravitational energy density” at a point, for any quantity with dimensions of energy constructed from the metric tensor g_{ab} (typically this involves expressions which are quadratic in the first derivatives of the metric tensor). This means that in General Relativity we cannot pursue the same procedure used to define *radiating fields* as in electromagnetism **if we insist upon using quantities with dimensions of energy for this purpose**. Of course, this does not imply that “gravitational energy” is meaningless and in fact we can construct quasilocal and global quantities with dimensions of energy which tell us when a gravitational system is radiating. This has been performed for the important case of isolated systems where the quantity is the Bondi mass [9, 37, 35].

If instead of energy, we use superenergy as a replacement, then the aforementioned problem disappears and one can use the same ideas as in electromagnetism to define *radiating gravitational fields* or *radiating spacetimes* in a local way. This approach was pioneered by Bel many years ago in [5] and, indeed, the results presented

in this section can be regarded as a continuation of Bel's work. We must bear in mind all the time that radiating gravitational fields defined in terms of superenergy are in principle different from radiating fields defined by a quasilocal energy prescription. To find the precise relation between both concepts is an interesting open question which is a particular case of a more general problem, namely, the possible relationship between superenergy and energy. This is a long standing question which has been already largely researched [6, 7, 24, 28, 29] (a fuller list of references about this subject can be found in [38]).

8.1. Superenergy radiative states for vacuum spacetimes

Let us start by reviewing Bel's work about the definition of a radiative spacetime. The starting point of Bel's study was the linearized form of (7.12). To obtain this form, we define a coordinate chart (t, x^i) , $i = 1, 2, 3$ in such a way that $\partial/\partial t$ is the unit timelike vector n^a and $\{\partial/\partial x^i\}$ are spacelike $\forall i$. Next we approximate the spatial covariant derivative by a covariant derivative compatible with the frame $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\}$, and ignore terms containing kinematical quantities. Under this approximation, equations (7.12)-(7.13) become

$$\frac{\partial W}{\partial t} + \sum_{i=1}^3 \frac{\partial \mathcal{P}^i}{\partial x^i} = 0, \quad \frac{\partial \mathcal{P}_i}{\partial t} - \sum_{j=1}^3 \frac{\partial t^j_i}{\partial x^j} = 0. \quad (8.1)$$

These equations can always be obtained at a given point p of the spacetime if we choose an observer n^a such that all its kinematical quantities vanish at p (such an observer always exists according to the equivalence principle). The first equation of (8.1) has the form of a typical conservation law. The vector \mathcal{P}^i is, according to this equation, the flux of W (superenergy flux) and whenever \mathcal{P}^i is zero we see that W does not change for the observer $\partial/\partial t$. According to proposition 5.2 the superenergy density W is zero if and only if C_{abcd} vanishes as well and besides W is always nonnegative. Therefore, it is possible to take W as a replacement for the missing concept of "energy density" of the gravitation and we may consider that the existence of a flux of superenergy for any observer is an indication of the intrinsic presence of gravitational radiation. These ideas led Bel to the following definition [5].

Definition 8.1 (State of intrinsic gravitational radiation, Bel 1962.) *We say that there is a state of intrinsic gravitational radiation at a point $p \in V$ of a vacuum spacetime if $\mathcal{P}_a = \mathcal{P}_a(n)$ does not vanish at p for any n^a .*

A well-known consequence of definition 8.1 is that Petrov types N, II and III are always radiative. To show this it is enough to recall that the condition $\mathcal{P}^a = 0$ entails (5.5) which can only be true for either type I or type D. Note that definition 8.1 does not say anything about the radiative character of Petrov types I and D and in fact a more general definition would be needed to decide the issue. To obtain a generalization of definition 8.1 is our next task.

To generalize definition 8.1 we need to use the full information coming from the orthogonal splitting of $\nabla_a T^a_{bcd} = 0$ and not just (7.12) which only contains part of this information. Theorem 7.3 contains all that is needed in our endeavour. If we wish to use the variation of superenergy as a tool to define radiative states then we need to find the evolution of a spatial tensor whose vanishing is equivalent to the absence of a gravitational field (in vacuum this is just the condition $C_{abcd} = 0$). Bel's definition is based on the scalar W but proposition (5.2) tells us that the tensor t_{ab} plays a

similar role (and besides W is not independent of t_{ab}). The propagation of t_{ab} is given by (7.10) and we see that the only term in this equation not affected by kinematical quantities (and hence intrinsic) is $D_a Q^a_{bc}$.

Definition 8.2 (Intrinsic superenergy radiative state in vacuum) *In a vacuum spacetime there exists an intrinsic superenergy radiative state at a point $p \in V$ if $Q_{abc}(n)$ does not vanish at p for any unit timelike normal n^a .*

Remark 8.1 We use the name *superenergy radiative state* instead of Bel's original name of *radiative state* in order to stress the fact that our definition is based on gravitational superenergy.

Note that there are more tensors which have the relevant properties of t_{ab} explained above and therefore we could use their propagation as the starting point for a definition of superenergy radiative state. The consequence of this is that definition 8.2 admits alternative but equivalent formulations. To see an example, consider the spatial tensor

$$W_{ab} \equiv E_{ac} E^c_b + B_{ac} B^c_b. \quad (8.2)$$

Clearly, $W^a_a = W$ and $W_{ab} = 0 \iff C_{abcd} = 0$. Moreover, for any spatial vector x^a , $W_{ab} x^a x^b$ is non-negative by inspection. We find that in terms of W_{ab} equation (7.10) takes the equivalent form

$$\begin{aligned} \mathcal{L}_n W_{cd} = & A^a (-h_{cd} \mathcal{P}_a + \frac{1}{2} h_{ad} \mathcal{P}_c + \frac{1}{2} h_{ac} \mathcal{P}_d + 2S_{cda}) - 4\varepsilon_{ab(d} W_{c)}^b \omega^a - \frac{4}{3} \theta W_{cd} + \\ & + \left(\frac{t_{cdab}}{2} + h_{ca} h_{db} W + 3h_{cd} W_{ab} \right) \sigma^{ab} + D_a S_{cd}^a, \end{aligned} \quad (8.3)$$

where

$$S_{cda} \equiv 2B_{b(d} E_{c)e} \varepsilon_a^{eb}. \quad (8.4)$$

In view of (8.3) we deduce that definition 8.2 can be formulated by replacing Q_{abc} with S_{abc} . In fact from (8.4) and (5.2) we deduce

$$S_{cda} = \frac{1}{2} (Q_{acd} - h_{cd} Q^b_{ba}), \quad Q_{acd} = 2(S_{cda} - h_{cd} S^b_{ba}),$$

from which we conclude that both S_{abc} and Q_{abc} contain the same information and thus they should be deemed equivalent. We may expect that any reasonable definition of a superenergy radiative state should be formulated in terms of a spatial tensor which is equivalent to Q_{abc} . Any such tensor can be regarded as the gravitational equivalent of electromagnetism's Poynting vector. The tensor S_{abc} seems to be the simplest choice and one may adopt it as the basic geometric object measuring "superenergy flux".

Another interesting aspect of (7.10)-(7.11) or (8.3), already pointed out in remark 7.2, is the fact that they are written in such a way that the couplings of the kinematical quantities to the different spatial parts of the decomposition of the Bel-Robinson tensor are apparent. In our present context these couplings could be interpreted as the effect on the superenergy radiation due to the acceleration, the expansion, the shear and the rotation. At this point it is instructive to compare equation (7.10) (or its equivalent (8.3)) with its electromagnetic counterpart which is (3.7). In the electromagnetic case we realize that the vorticity has no effect whatsoever on the radiation of electromagnetic energy whereas it certainly influences the radiation of superenergy because ω^a (or equivalently ω_{ab}) appears explicitly in (7.10).

8.2. Superenergy radiative states for general spacetimes

Using the ideas explained in the previous section we can formulate a definition of an intrinsic superenergy radiative state that is similar to definition 8.2 but valid for a general spacetime. In this case we need to study the evolution of a spatial quantity which is zero if and only if the Riemann tensor vanishes. As stated in proposition 6.2 the tensor \bar{t}_{ab} has the required properties and hence the terms appearing in the evolution equation of \bar{t}_{ab} should enable us to define the concept of an intrinsic radiative state. The evolution equation sought is (7.3) and hence the inspection of this equation leads us to the following

Definition 8.3 (Intrinsic superenergy radiative state in a general spacetime)

There exists an intrinsic superenergy radiative state at a point $p \in V$ if for any unit timelike vector n^a it is the case that $\bar{Q}_{abc}(n)$ does not vanish at p .

Similar considerations as in the case of definition 8.2 apply here.

9. Conclusions and open issues

In this work we have obtained the full orthogonal splitting of the Bel tensor and its covariant divergence and we have particularized it to the important case of vacuum spacetimes where the Bel tensor becomes the Bel-Robinson tensor. This gives rise to the dynamical laws of superenergy. The concept of a superenergy radiative state has been introduced. The work just presented opens new research lines which we believe are worth exploring. Perhaps one of the most interesting issues is a *global* formulation of the dynamical laws of superenergy complementing the local formulation of theorem 7.1. Such a global formulation would enable us to apply our techniques to realistic astrophysical settings such as oscillating stars, rotating bodies or radiating binary systems.

In this paper we have restricted ourselves to the superenergy defined from the Riemann and Weyl tensor but one can define tensors representing superenergy from a general field resulting in the *superenergy tensor* of that field [38]. In this framework it is possible to calculate the covariant divergence of a superenergy tensor and obtain an expression similar to the first equation in (4.17) with the Bel tensor replaced by a suitable superenergy tensor. The orthogonal splitting of such an equation would yield the dynamical laws of the superenergy associated with that particular field. An interesting example concerns the electromagnetic field. In this case a possible superenergy tensor is the Chevreton tensor which was first introduced in [15] and recently stimulating results about its symmetries and the covariant divergence of its trace have been obtained [8]. The Chevreton tensor, like the Bel-Robinson tensor, is a rank-four tensor and its covariant divergence couples the Weyl tensor with terms which contain covariant derivatives of the Faraday tensor [18]. This suggests a possible exchange between the gravitational and the electromagnetic superenergies [38, 30, 19]. The orthogonal splitting of the covariant divergence of the Chevreton tensor might shed light on the nature of this exchange.

Another important issue is the possible relationship between superenergy and any of the available quasilocal concepts of *gravitational energy* which have been developed over the years. This is a topic which has been extensively researched in the past and no clear conclusion has been reached. In this work no attempt has been made in this direction and our point of view has been to regard superenergy as a physical

quantity on its own right. We believe that this idea can be put to work by means of the results of theorem 7.1 which would demand a formulation of the dynamical laws of superenergy tailored for each physical system under study. However, a relation between superenergy and gravitational energy cannot be ruled out and the orthogonal splitting of the Bel tensor might bring a new point of view to this old problem.

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Appendix A. Technical details about the computations

In this appendix we supply details about the calculations required in this work. In order to do so we need to explain some implementation aspects of the system *xAct*. We will limit ourselves to only those issues which are needed in our calculations referring the interested reader to [33] for a full documentation and tutorials about *xAct*.

Orthogonal splittings play an essential part in our work and the implementation of *xAct* in regard to this matter is completely adapted to our requirements. The basic elements of the orthogonal splitting are defined through the command

```
In[]:= DefMetric[1, h[-a, -b], cd, {"|", "D"}, InducedFrom->{g, n}, PrintAs->"h"].
```

Here $h[-a, -b]$ represents the spatial metric h_{ab} which is constructed from the spacetime metric g_{ab} (represented in the system by $g[-a, -b]$) and the unit normal vector n^a (represented by $n[a]$). The operator $cd[-a]$ is the Cattaneo operator D_a associated with h_{ab} . The system is able to handle all the properties of the Cattaneo operator explained in subsection 2.2 in a natural fashion.

The general expression for the orthogonal splitting of any tensor is equation (2.3). This result is implemented in *xAct* by means of the command

```
In[]:= InducedDecomposition[expr, {h,n}],
```

where **expr** represents any tensorial expression. The output of **InducedDecomposition** is the result of applying formula (2.3) to **expr**. The orthogonal projector operator P_h which appears in (2.3) is also implemented in *xAct* by means of the command **Projectorh[expr]** where again **expr** represents an arbitrary tensor. The basic commands just explained enable us to find efficiently orthogonal splittings similar to equation (2.12) with L_a replaced by any spatial tensor of higher rank.

Proof of theorems 7.1 and 7.2

To prove theorems 7.1 and 7.2 we need to find the orthogonal decomposition of the equations shown in (4.17). The first step is to replace B_{abcd} , R_{abcd} and \mathfrak{J}_{abc} with their orthogonal splittings, eqs. (6.1), (4.6) and (4.10) respectively. The covariant

derivatives of n^a are decomposed according to (2.4) and $\nabla_a \varepsilon_{bcd}$ is decomposed by means of the formula

$$\nabla_d \varepsilon_{fhl} = 3A^a n_d n_{[h} \varepsilon_{f]l]a} - 3n_{[h} \left(\frac{1}{3} \varepsilon_{f]l]d} \theta + \varepsilon_{f]l]a} (\sigma_d^a + \omega_d^a) \right).$$

After doing these replacements we obtain expressions which contain $\nabla_a \overline{W}$, $\nabla_a \overline{\mathcal{P}}_b$, $\nabla_a \overline{t}_{bc}$, $\nabla_a t_{bc}^*$, $\nabla_a \overline{t}_{bcde}$, $\nabla_a L_b$, $\nabla_a \tilde{J}_{bc}$, $\nabla_a \overline{J}_{bc}$, $\nabla_a j_{bcd}$. These are further decomposed by following the procedure explained in subsection 2.2. For example, the orthogonal decomposition of $\nabla_a L_b$ is just equation (2.12) which also holds if we replace L_b with $\overline{\mathcal{P}}_b$. Other orthogonal decompositions needed are

$$\begin{aligned} \nabla_c \overline{t}_{ab} = & -2A^d n_c n_{(b} \overline{t}_{a)d} + 2n_{(a} \left(\frac{1}{3} \overline{t}_{b)c} \theta + \overline{t}_{b)}^d (\sigma_{cd} + \omega_{cd}) \right) + D_c \overline{t}_{ab} + \\ & + n_c \left(\frac{2}{3} \theta \overline{t}_{ab} + 2\overline{t}_{(b}^d (\sigma_{a)d} + \omega_{a)d}) - \mathcal{L}_n \overline{t}_{ab} \right), \end{aligned} \quad (\text{A.1})$$

which is also valid if we replace \overline{t}_{ab} by any symmetric spatial tensor and

$$\begin{aligned} \nabla_a \tilde{J}_{bc} = & 2A^d n_a (\tilde{J}_{d[b} n_{c]}) + 2n_{[c} \left(\frac{1}{3} \tilde{J}_{b]a} \theta + \tilde{J}_{b]}^d (\sigma_{ad} + \omega_{ad}) \right) + D_a \tilde{J}_{bc} + \\ & + n_a \left(\frac{2}{3} \tilde{J}_{bc} \theta + 2\tilde{J}_{[b}^c (\sigma_{c]d} + \omega_{c]d}) - \mathcal{L}_n \tilde{J}_{bc} \right), \end{aligned} \quad (\text{A.2})$$

which is true if we replace \tilde{J}_{bc} with any antisymmetric tensor. The expressions for the orthogonal splitting of the remaining covariant derivatives are very long and we omit them. Inserting the orthogonal splittings in (4.17) and rearranging the equations obtained as polynomials in n^a we obtain the expressions

$$An^b n^c n^d + B_1^b n^c n^d + B_2^{(c} n^d) n^b + C_1^{cd} n^b + C_2^{b(c} n^d) + E^{bcd} = 0, \quad (\text{A.3})$$

$$G_1^{[b} n^{c]} + I^{bc} = 0, \quad (\text{A.4})$$

where all the tensor coefficients of these polynomials are spatial. This implies that the coefficients of the polynomials must vanish and these conditions lead us to

$$A = 0, B_1^b = B_2^b = 0, C_1^{(cd)} = C_1^{cd} = 0, C_2^{bc} = 0, E^{bcd} = 0, G_1^b = 0, I^{[bc]} = I^{bc} = 0. \quad (\text{A.5})$$

After some manipulations we find that the condition $A = 0$ is equivalent to (7.9), $B_1^b = B_2^b = 0$ are equivalent to (7.1)-(7.2), $C_1^{cd} = 0$ is equivalent to (7.3), $C_2^{bc} = 0$ is equivalent to (7.4), $E^{bcd} = 0$ is equivalent to (7.5), $G_1^b = 0$ is equivalent to (7.7) and $I^{bc} = 0$ is equivalent to (7.8). The condition $A = 0$ is redundant because it can be obtained as the trace of $C_1^{cd} = 0$ and therefore we do not need to consider it. We have thus recovered all the expressions given in the statements of theorems 7.1 and 7.2. Note that the polynomials (A.3)-(A.4) are equivalent to each of the equations presented in (4.17) and so is the set of conditions stemming from (A.5). \square

Appendix B. Canonical forms for the electric and magnetic parts of Weyl tensor in the different Petrov types.

We present next the canonical forms of the electric and magnetic parts of Weyl tensor for the different Petrov types. We follow [5] in our presentation (see also [40] for an equivalent representation of the canonical forms). All the canonical forms are written with respect to a certain orthonormal frame $O \equiv \{e_1^a, e_2^a, e_3^a\}$ of spatial vectors (canonical frame).

Petrov type I

In this type E_{ab} and B_{ab} take the following form in the canonical frame O

$$E_{\mathbf{ab}} = \text{diag}(E_{11}, E_{22}, E_{33}), \quad B_{\mathbf{ab}} = \text{diag}(B_{11}, B_{22}, B_{33}), \quad (\text{B.1})$$

with the additional conditions

$$E_{11} + E_{22} + E_{33} = 0, \quad B_{11} + B_{22} + B_{33} = 0. \quad (\text{B.2})$$

Petrov type D

This type arises if we set $-\frac{1}{2}E_{11} = E_{22} = E_{33}$, $-\frac{1}{2}B_{11} = B_{22} = B_{33}$ in the previous case.

Petrov type II

The canonical forms for E_{ab} , B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & -\frac{E_{11}}{2} + B_{23} & E_{23} \\ 0 & E_{23} & -\frac{E_{11}}{2} - B_{23} \end{pmatrix},$$

$$B_{\mathbf{ab}} = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & -\frac{B_{11}}{2} - E_{23} & B_{23} \\ 0 & B_{23} & -\frac{B_{11}}{2} + E_{23} \end{pmatrix}. \quad (\text{B.3})$$

Petrov type III

The canonical forms for E_{ab} , B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} 0 & E_{12} & -B_{12} \\ E_{12} & 0 & 0 \\ -B_{12} & 0 & 0 \end{pmatrix}, \quad B_{\mathbf{ab}} = \begin{pmatrix} 0 & B_{12} & E_{12} \\ B_{12} & 0 & 0 \\ E_{12} & 0 & 0 \end{pmatrix}. \quad (\text{B.4})$$

Petrov type N

The canonical forms for E_{ab} , B_{ab} in the frame O are

$$E_{\mathbf{ab}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{22} & -B_{22} \\ 0 & -B_{22} & -E_{22} \end{pmatrix}, \quad B_{\mathbf{ab}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{22} & E_{22} \\ 0 & E_{22} & -B_{22} \end{pmatrix}. \quad (\text{B.5})$$

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